

# EXISTENCE OF AN ISOLATED BAND IN A SYSTEM OF THREE PARTICLES IN AN OPTICAL LATTICE

GIANFAUSTO DELL'ANTONIO, SAIDAKHMAT N. LAKAEV, AHMAD M. KHALKHUZHAEV

**ABSTRACT.** We prove the existence of two-and three-particle bound states of the Schrödinger operators  $h_\mu(k), k \in \mathbb{T}^d$  and  $H_\mu(K), K \in \mathbb{T}^d$  associated to Hamiltonians  $h_\mu$  and  $H_\mu$  of a system of two and three identical bosons on the lattice  $\mathbb{Z}^d, d = 1, 2$  interacting via pairwise zero-range attractive  $\mu < 0$  or repulsive  $\mu > 0$  potentials. As a consequence we show the existence of an isolated band in the two and three bosonic systems in an optical lattice.

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## 1. INTRODUCTION

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions that arises from the lattice band structure [9].

The Bose-Hubbard model which is used to describe the repulsive pairs is the theoretical basis for applications. The work [9] exemplifies the important correspondence between the Bose-Hubbard model [3, 4] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications.

The main goal of the paper is to prove the existence of three-bosonic bound states of the Schrödinger operator  $H_\mu(K), K \in \mathbb{T}^d$  for the cases of attractive  $\mu < 0$  and repulsive  $\mu > 0$  interactions.

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms.

The dynamics of the ultracold atoms loaded in the lower or upper band is well described by the Bose-Hubbard hamiltonian [9]; we give in section 2 the corresponding Schrödinger operator.

In the continuum case due to rotational invariance the Hamiltonian separates in a free hamiltonian for the center of mass and in a hamiltonian  $H_{\text{rel}}$  for the relative motion. Bound states are eigenstates of  $H_{\text{rel}}$ .

The kinematics of the quantum particles on the lattice is rather exotic. The discrete laplacian is *not rotationally invariant* and therefore one cannot separate the motion of the center of mass.

One can rather resort to a Floquet-Bloch decomposition. The three-particle Hilbert space  $\mathcal{H} \equiv \ell^2[(\mathbb{Z}^d)^3]$  is represented as direct integral associated to the representation of

the discrete group  $\mathbb{Z}^d$  by shift operators

$$\ell^2[(\mathbb{Z}^d)^3] = \int_{K \in \mathbb{T}^d} \oplus \ell^2[(\mathbb{Z}^d)^2] \eta(dK),$$

where  $\eta(dp) = \frac{d^d p}{(2\pi)^d}$  is the (normalized) Haar measure on the torus  $\mathbb{T}^d$ . Hence the total three-body hamiltonian appears to be decomposable

$$H = \int_{\mathbb{T}^d} \oplus H(K) \eta(dK).$$

The fiber hamiltonian  $H(K)$  depends perimetrically on the *quasi momentum*  $K \in \mathbb{T}^d \equiv \mathbb{R}^d / (2\pi\mathbb{Z}^d)$ . It is the sum of a free part and an interaction term, both bounded and the dependence on  $K$  of the free part is continuous.

Bound states  $\psi_{E,K}$  are solution of the Schrödinger equation

$$H(K)\psi_{E,K} = E\psi_{E,K}, \quad \psi_{E,K} \in \ell^2[(\mathbb{Z}^d)^2].$$

It is known that in dimension  $d = 3$  for the case we are considering the hamiltonian  $H(K)$ ,  $K \in \mathbb{T}^3$  for  $K = 0 \in \mathbb{T}^3$  has infinitely many bound states (the Efimov effect)[1, 6]. Since, the operator  $H(K)$ ,  $K \in \mathbb{T}^3$  continuously depends on  $K \in \mathbb{T}^3$  one can conclude that there exists a neighborhood  $\mathbb{G}_0 \subset \mathbb{T}^3$  of  $0 \in \mathbb{T}^3$  and for all  $K \in \mathbb{G}_0$  the operator  $H(K)$ ,  $K \in \mathbb{G}_0$  has bound states [2, 6].

In the paper we study the hamiltonian  $H_\mu$  of a system of three bosons on the lattice  $\mathbb{Z}^d$ ,  $d = 1, 2$  interacting through attractive  $\mu < 0$  or repulsive  $\mu > 0$  zero-range potential  $\mu V$ .

Remark that the  $d = 1, 2$  situation is of interest, since in the experiment it corresponds to a low depth of the lattice along one direction, whilst the lattice in the perpendicular directions remains very deep [9].

Our main new results is to prove the existence of three-bosonic bound states of the Scarödinger operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$  associated to  $H_\mu$ , with energy lying below the bottom resp. above the top of the essential spectrum for the case of attractive  $\mu < 0$  resp. repulsive  $\mu > 0$  interaction.

As a consequence we show the existence of a band spectrum of the hamiltonian  $H_\mu$  of a system of three bosons.

We can conclude that these results for a three bosonic system theoretically predicts the existence stable attractively and repulsively bound objects of three atoms. Hopefully, this also can be experimentally confirmed as it done for atoms pair with repulsive interaction in [9].

To our knowledge analogous results have not been published yet even for a system of three particles interacting via attractive potentials on Euclidean space  $\mathbb{R}^d$ .

We remark that the same formalism could be used to prove the existence of at least one bound state with energy lying below resp. above the essential spectrum for the case of attractive resp. repulsive short range potentials.

This paper is organized as follows.

Section 1 is introduction. In section 2 we give explicitly the hamiltonian of the two-body and three-body case in the Schrödinger representation. It corresponds to the Hubbard hamiltonian in the number of particles representation. In section 3 we introduce the Floquet-Bloch decomposition (von Neumann decomposition) and choose relative coordinates to describe explicitly the discrete Schrödinger operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$ . In section 4 we state our main results. In section 5 we introduce *channel operators* and describe the

essential spectrum of  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$  by means discrete spectrum of  $h_\mu(k)$ ,  $k \in \mathbb{T}^d$ . We prove the existence of bound states in section 6.

## 2. HAMILTONIANS OF THREE IDENTICAL BOSONS ON LATTICES IN THE COORDINATE AND MOMENTUM REPRESENTATIONS

Let  $\mathbb{Z}^d, d = 1, 2$  be the  $d$ -dimensional lattice. Let  $\ell^2[(\mathbb{Z}^d)^m], d = 1, 2$  be Hilbert space of square-summable functions  $\hat{\varphi}$  defined on the Cartesian power  $(\mathbb{Z}^d)^m, d = 1, 2$  and let  $\ell^{2,s}[(\mathbb{Z}^d)^m] \subset \ell^2[(\mathbb{Z}^d)^m]$  be the subspace of functions symmetric with respect to the permutation of coordinates of the particles.

Let  $\Delta$  be the lattice Laplacian, i.e., the operator which describes the transport of a particle from one site to another site:

$$(\Delta\hat{\psi})(x) = - \sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x+s)], \quad \hat{\psi} \in \ell^2(\mathbb{Z}^d).$$

The free hamiltonian  $\hat{h}_0$  of a system of two identical quantum mechanical particles with mass  $m = 1$  on the  $d$ -dimensional lattice  $\mathbb{Z}^d, d = 1, 2$  in the coordinate representation is associated to the self-adjoint operator  $\hat{h}_0$  in the Hilbert space  $\ell^{2,s}[(\mathbb{Z}^d)^2]$ :

$$\hat{h}_0 = \Delta \otimes I + I \otimes \Delta.$$

The total hamiltonian  $\hat{h}_\mu$  of a system of two quantum-mechanical identical particles with the two-particle pairwise zero-range attractive interaction  $\mu\hat{v}$  is a bounded perturbation of the free hamiltonian  $\hat{h}_0$  on the Hilbert space  $\ell^{2,s}[(\mathbb{Z}^d)^2]$ :

$$\hat{h}_\mu = \hat{h}_0 + \mu\hat{v}.$$

Here  $\mu \in \mathbb{R}$  is coupling constant and

$$(\hat{v}\hat{\psi})(x_1, x_2) = \delta_{x_1 x_2} \hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^{2,s}[(\mathbb{Z}^d)^2],$$

where  $\delta_{x_1 x_2}$  is the Frolicker delta.

Analogously, the free hamiltonian  $\hat{H}_0$  of a system of three identical bosons with mass  $m = 1$  on the  $d$ -dimensional lattice  $\mathbb{Z}^d$  is defined on  $\ell^{2,s}[(\mathbb{Z}^d)^3]$ :

$$\hat{H}_0 = \Delta \otimes I \otimes I + I \otimes \Delta \otimes I + I \otimes I \otimes \Delta.$$

The total hamiltonian  $\hat{H}_\mu$  of a system of three quantum-mechanical identical particles with pairwise zero-range interaction  $\hat{v} = \hat{v}_\alpha = \hat{v}_{\beta\gamma}, \alpha, \beta, \gamma = 1, 2, 3$  is a bounded perturbation of the free hamiltonian  $\hat{H}_0$ :

$$\hat{H}_\mu = \hat{H}_0 + \mu\hat{V},$$

where  $\hat{V} = \sum_{i=1}^3 \hat{V}_\alpha, V_\alpha = \hat{V}, \alpha = 1, 2, 3$  is the multiplication operator on  $\ell^{2,s}[(\mathbb{Z}^d)^3]$  defined by

$$(\hat{V}_\alpha \hat{\psi})(x_1, x_2, x_3) = \delta_{x_\beta x_\gamma} \hat{\psi}(x_1, x_2, x_3), \\ \alpha \prec \beta \prec \gamma \prec \alpha, \alpha, \beta, \gamma = 1, 2, 3, \hat{\psi} \in \ell^{2,s}[(\mathbb{Z}^d)^3].$$

**2.1. The momentum representation.** Let  $\mathbb{T}^d = (-\pi, \pi]^d$  be the  $d$ -dimensional torus and  $L^{2,s}[(\mathbb{T}^d)^m] \subset L^2[(\mathbb{T}^d)^m]$  be the subspace of symmetric functions defined on the Cartesian power  $(\mathbb{T}^d)^m$ ,  $m \in \mathbb{N}$ .

Let

$$\hat{\Delta} = \mathcal{F}\Delta\mathcal{F}^*$$

be Fourier transform of the Laplacian  $\Delta$ , where

$$\mathcal{F} : \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{T}^d), \quad [\mathcal{F}(f)](p) := \sum_{x \in \mathbb{Z}^d} e^{-i(p,x)} f(x)$$

is the standard Fourier transform with the inverse

$$\mathcal{F}^* : L^2(\mathbb{T}^d) \rightarrow \ell^2(\mathbb{Z}^d), \quad [\mathcal{F}^*(\psi)](x) := \int_{\mathbb{T}^d} e^{i(p,x)} \psi(p) \eta(dp),$$

and  $\eta(dp) = \frac{d^d p}{(2\pi)^d}$  is the (normalized) Haar measure on the torus. It easily checked that  $\hat{\Delta}$  is the multiplication operator by the function  $\varepsilon(\cdot)$ , i.e.

$$(\hat{\Delta}f)(k) = \varepsilon(p)f(p), \quad f \in L^2(\mathbb{T}^d),$$

where

$$\varepsilon(p) = 2 \sum_{i=1}^d (1 - \cos p^{(i)}), \quad p = (p^{(1)}, \dots, p^{(d)}) \in \mathbb{T}^d.$$

The two-particle total hamiltonian  $h_\mu$  in the momentum representation is given on  $L^{2,s}[(\mathbb{T}^d)^2]$  as follows

$$h_\mu = h_0 + \mu v.$$

Here the free hamiltonian  $h_0$  is of the form

$$h_0 = \hat{\Delta} \otimes I + I \otimes \hat{\Delta},$$

where  $I$  is identity operator on  $L^2(\mathbb{T}^d)$  and  $\otimes$  denotes the tensor product. It is easy to see that the operator  $h_0$  is the multiplication operator by the function  $\varepsilon(k_1) + \varepsilon(k_2)$  :

$$(h_0 f)(k_1, k_2) = [\varepsilon(k_1) + \varepsilon(k_2)]f(k_1, k_2), \quad f \in L^{2,s}[(\mathbb{T}^d)^2],$$

The integral operator  $v$  is the convolution type

$$\begin{aligned} (vf)(k_1, k_2) &= \int_{(\mathbb{T}^d)^2} \delta(k_1 + k_2 - k'_1 - k'_2) f(k'_1, k'_2) \eta(dk'_1) \eta(dk'_2) \\ &= \int_{\mathbb{T}^d} f(k'_1, k_1 + k_2 - k'_1) \eta(dk'_1), \quad f \in L^{2,s}[(\mathbb{T}^d)^2], \end{aligned}$$

where  $\delta(\cdot)$  is the  $d$ -dimensional Dirac delta function.

The three-particle hamiltonian in the momentum representation is given by the bounded self-adjoint operator on the Hilbert space  $L^{2,s}[(\mathbb{T}^d)^3]$  as follows

$$H_\mu = H_0 + \mu(V_1 + V_2 + V_3),$$

where  $H_0$  is of the form

$$H_0 = \hat{\Delta} \otimes I \otimes I + I \otimes \hat{\Delta} \otimes I + I \otimes I \otimes \hat{\Delta},$$

i.e., the free hamiltonian  $H_0$  is the multiplication operator by the function  $\sum_{\alpha=1}^3 \varepsilon(k_\alpha)$ :

$$(H_0 f)(k_1, k_2, k_3) = \left[ \sum_{\alpha=1}^3 \varepsilon(k_\alpha) \right] f(k_1, k_2, k_3),$$

and

$$\begin{aligned} & (V_\alpha f)(k_\alpha, k_\beta, k_\gamma) \\ &= \int_{(\mathbb{T}^d)^3} \delta(k_\alpha - k'_\alpha) \delta(k_\beta + k_\gamma - k'_\alpha - k'_\beta) f(k'_1, k'_2, k'_3) \eta(dk'_\alpha) \eta(dk'_\beta) \eta(dk'_\gamma) \\ &= \int_{\mathbb{T}^d} f(k_\alpha, k'_\beta, k_\beta + k_\gamma - k'_\beta) \eta(dk'_\beta), \quad f \in L^{2,s}[(\mathbb{T}^d)^3]. \end{aligned}$$

### 3. DECOMPOSITION OF THE HAMILTONIANS INTO VON NEUMANN DIRECT INTEGRALS. QUASI-MOMENTUM AND COORDINATE SYSTEMS

Denote by  $k = k_1 + k_2 \in \mathbb{T}^d$  resp.  $K = k_1 + k_2 + k_3 \in \mathbb{T}^d$  the *two-* resp. *three-particle quasi-momentum* and define the set  $\mathbb{Q}_k$  resp.  $\mathbb{Q}_K$  as follows

$$\mathbb{Q}_k = \{(k_1, k - k_1) \in (\mathbb{T}^d)^2 : k_1 \in \mathbb{T}^d, k - k_1 \in \mathbb{T}^d\}$$

resp.

$$\mathbb{Q}_K = \{(k_1, k_2, K - k_1 - k_2) \in (\mathbb{T}^d)^3 : k_1, k_2 \in \mathbb{T}^d, K - k_1 - k_2 \in \mathbb{T}^d\}.$$

We introduce the mapping

$$\pi_1 : (\mathbb{T}^d)^2 \rightarrow \mathbb{T}^d, \quad \pi_1(k_1, k_2) = k_1$$

resp.

$$\pi_2 : (\mathbb{T}^d)^3 \rightarrow (\mathbb{T}^d)^2, \quad \pi_2(k_1, k_2, k_3) = (k_1, k_2).$$

Denote by  $\pi_k, k \in \mathbb{T}^d$  resp.  $\pi_K, K \in \mathbb{T}^d$  the restriction of  $\pi_1$  resp.  $\pi_2$  onto  $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$ , resp.  $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$ , that is,

$$\pi_k = \pi_1|_{\mathbb{Q}_k} \quad \text{and} \quad \pi_K = \pi_2|_{\mathbb{Q}_K}.$$

At this point it is useful to remark that  $\mathbb{Q}_k, k \in \mathbb{T}^d$  resp.  $\mathbb{Q}_K, K \in \mathbb{T}^d$  are  $d$ - resp.  $2d$ -dimensional manifold isomorphic to  $\mathbb{T}^d$  resp.  $(\mathbb{T}^d)^2$ .

**Lemma 3.1.** *The map  $\pi_k, k \in \mathbb{T}^d$  resp.  $\pi_K, K \in \mathbb{T}^d$  is bijective from  $\mathbb{Q}_k \subset (\mathbb{T}^d)^2$  resp.  $\mathbb{Q}_K \subset (\mathbb{T}^d)^3$  onto  $\mathbb{T}^d$  resp.  $(\mathbb{T}^d)^2$  with the inverse map given by*

$$(\pi_k)^{-1}(k_1) = (k_1, k - k_1)$$

resp.

$$(\pi_K)^{-1}(k_1, k_2) = (k_1, k_2, K - k_1 - k_2).$$

Let  $L^{2,e}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$  be the subspace of even functions. Decomposing the Hilbert space  $L^{2,s}[(\mathbb{T}^d)^2]$  resp.  $L^{2,s}[(\mathbb{T}^d)^3]$  into the direct integral

$$L^{2,s}[(\mathbb{T}^d)^2] = \int_{k \in \mathbb{T}^d} \oplus L^{2,e}(\mathbb{T}^d) \eta(dk)$$

resp.

$$L^{2,s}[(\mathbb{T}^d)^3] = \int_{K \in \mathbb{T}^d} \oplus L^{2,s}[(\mathbb{T}^d)^2] \eta(dK)$$

yields the decomposition of the hamiltonian  $h_\mu$  resp.  $H_\mu$  into the direct integral

$$(3.1) \quad h_\mu = \int_{k \in \mathbb{T}^d} \oplus \tilde{h}_\mu(k) \eta(dk)$$

resp.

$$(3.2) \quad H_\mu = \int_{K \in \mathbb{T}^d} \oplus \tilde{H}_\mu(K) \eta(dK).$$

**3.1. The discrete Schrödinger operators.** The fiber operator  $\tilde{h}_\mu(k)$ ,  $k \in \mathbb{T}^d$  from the direct integral decomposition (3.1) acting in  $L^2[\mathbb{Q}_k]$  according to Lemma 3.1 is unitarily equivalent to the operator  $h_\mu(k)$ ,  $k \in \mathbb{T}^d$  given by

$$(3.3) \quad h_\mu(k) = h_0(k) + \mu v.$$

The operator  $h_0(k)$  is multiplication operator by the function  $\mathcal{E}_k(p)$ :

$$(h_0(k)f)(p) = \mathcal{E}_k(p)f(p), \quad f \in L^{2,e}(\mathbb{T}^d),$$

where

$$\mathcal{E}_k(p) = \varepsilon(k - p) + \varepsilon(p)$$

and

$$(vf)(p) = \int_{\mathbb{T}^d} f(q) d\eta(q), \quad f \in L^{2,e}(\mathbb{T}^d).$$

The fiber operator  $\tilde{H}_\mu(K)$ ,  $K \in \mathbb{T}^d$  from the direct integral decomposition (3.2) acting in  $L^2[\mathbb{Q}_K]$  according to Lemma 3.1 is unitarily equivalent to the operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$  given by

$$H_\mu(K) = H_0(K) + \mu(V_1 + V_2 + V_3).$$

The operator  $H_0(K)$  acts in the Hilbert space  $L^{2,s}[(\mathbb{T}^d)^2]$  and has form

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L^{2,s}[(\mathbb{T}^d)^2],$$

where

$$E(K; p, q) = \varepsilon(K - p - q) + \varepsilon(q) + \varepsilon(p).$$

The operator  $\mathbb{V} = V_1 + V_2 + V_3$  acting on  $L^{2,s}[(\mathbb{T}^d)^2]$  in coordinates  $(p, q) \in (\mathbb{T}^d)^2$  can be written in the form

$$(\mathbb{V}f)(p, q) = \int_{\mathbb{T}^d} f(p, t) \eta(dt) + \int_{\mathbb{T}^d} f(t, q) \eta(dt) + \int_{\mathbb{T}^d} f(t, K - p - q) \eta(dt).$$

#### 4. STATEMENT OF THE MAIN RESULTS

According to the Weyl theorem [8] the essential spectrum  $\sigma_{\text{ess}}(h_\mu(k))$  of the operator  $h_\mu(k)$ ,  $k \in \mathbb{T}^d$  coincides with the spectrum  $\sigma(h_0(k))$  of  $h_0(k)$ . More specifically,

$$\sigma_{\text{ess}}(h_\mu(k)) = [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)],$$

where

$$\begin{aligned} \mathcal{E}_{\min}(k) &\equiv \min_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2 \sum_{i=1}^d [1 - \cos(\frac{k^{(i)}}{2})] \\ \mathcal{E}_{\max}(k) &\equiv \max_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2 \sum_{i=1}^d [1 + \cos(\frac{k^{(i)}}{2})]. \end{aligned}$$

Since for each  $K \in \mathbb{T}^d$  the function  $E(K; p, q)$  is continuous on  $(\mathbb{T}^d)^2$ ,  $d = 1, 2$  there exist

$$E_{\min}(K) \equiv \min_{p, q \in \mathbb{T}^d} E(K; p, q), \quad E_{\max}(K) \equiv \max_{p, q \in \mathbb{T}^d} E(K; p, q)$$

and  $\sigma(H_0(K)) = [E_{\min}(K), E_{\max}(K)]$ .

**Remark 4.1.** We remark that the essential spectrum  $[\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)]$  strongly depends on the quasi-momentum  $k \in \mathbb{T}^d$ ; when  $k = \vec{\pi} = (\pi, \dots, \pi) \in \mathbb{T}^d$  the essential spectrum of  $h_\mu(k)$  degenerated to the set consisting of a unique point  $\{\mathcal{E}_{\min}(\vec{\pi}) = \mathcal{E}_{\max}(\vec{\pi}) = 2d\}$  and hence the essential spectrum of  $h_\mu(k)$  is not absolutely continuous for all  $k \in \mathbb{T}^d$ . The similar arguments are true for the essential spectrum of  $H_0(K)$ .

The following theorem asserts the existence of eigenvalues of the operator  $h_\mu(k)$  and can be proven in the same way as Theorem 4.2 in [5, 7].

**Theorem 4.2.** For any  $\mu < 0$  resp.  $\mu > 0$  and  $k \in \mathbb{T}^d$ ,  $d = 1, 2$  the operator  $h_\mu(k)$  has a unique eigenvalue  $e_\mu(k)$ , which is even in  $k \in \mathbb{T}^d$  and satisfies the relations:

$$e_\mu(k) < \mathcal{E}_{\min}(k), \quad k \in \mathbb{T}^d \text{ and } e_\mu(0) < e_\mu(k), \quad k \in \mathbb{T}^d \setminus \{0\} \text{ for } \mu < 0$$

resp.

$$e_\mu(k) > \mathcal{E}_{\max}(k), \quad k \in \mathbb{T}^d \text{ and } e_\mu(0) > e_\mu(k), \quad k \in \mathbb{T}^d \setminus \{0\} \text{ for } \mu > 0.$$

The eigenvalue  $e_\mu(k)$  is holomorphic function in  $k \in \mathbb{T}^d$ . For any  $k \in \mathbb{T}^d$  the associated eigenfunction  $f_{\mu, e_\mu(k)}(p)$  is holomorphic in  $p \in \mathbb{T}^d$  and has the form

$$f_{\mu, e_\mu(k)}(\cdot) = \frac{\mu c(k)}{\mathcal{E}_k(\cdot) - e_\mu(k)}, \quad \mu < 0, \text{ resp. } f_{\mu, e_\mu(k)}(\cdot) = \frac{\mu c(k)}{e_\mu(k) - \mathcal{E}_k(\cdot)}, \quad \mu > 0$$

where  $c(k) \neq 0$  is a normalizing constant. Moreover, the vector valued mapping

$$f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dk); L^2(\mathbb{T}^d)], \quad k \rightarrow f_{\mu, e_\mu}, \quad \mu < 0$$

resp.

$$f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dk); L^2(\mathbb{T}^d)], \quad k \rightarrow f_{\mu, e_\mu}, \quad \mu > 0$$

is holomorphic on  $\mathbb{T}^d$ .

In the next theorem the essential spectrum of the three-particle operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$  is described by the spectra of the non perturbed operator  $H_0(K)$  and the discrete spectrum of the two-particle operator  $h_\mu(k)$ ,  $k \in \mathbb{T}^d$ .

**Theorem 4.3.** Let  $d = 1, 2$ . For any  $\mu \neq 0$  the essential spectrum  $\sigma_{\text{ess}}(H_\mu(K))$  of  $H_\mu(K)$  satisfies the following equality

$$\sigma_{\text{ess}}(H_\mu(K)) = \cup_{k \in \mathbb{T}^d} \{e_\mu(K - k) + \varepsilon(k)\} \cup [E_{\min}(K), E_{\max}(K)],$$

where  $e_\mu(k)$  is the unique eigenvalue of the operator  $h_\mu(k)$ ,  $k \in \mathbb{T}^d$ .

The main result of the paper is given in following theorem, which states the existence of bound states of the three-particle operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^d$ .

**Theorem 4.4.** Let  $d = 1, 2$ .

- (i) For all  $\mu < 0$  and  $K \in \mathbb{T}^d$  the operator  $H_\mu(K)$  has an eigenvalue  $E_\mu(K)$  lying below the bottom  $\tau_{\text{ess}}^b(H_\mu(K))$  of the essential spectrum. The eigenvalue  $E_\mu(K)$  is a holomorphic function in  $K \in \mathbb{T}^d$ . Moreover, the associated eigenfunction (bound state)  $f_{\mu, E_\mu(K)}(\cdot, \cdot) \in L^{2,s}[(\mathbb{T}^d)^2]$  is holomorphic in  $(p, q) \in (\mathbb{T}^d)^2$  and the vector valued mapping

$$f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dK); L^{2,s}[(\mathbb{T}^d)^2]], k \rightarrow f_{\mu, E_\mu(K)}$$

is also holomorphic in  $K \in \mathbb{T}^d$ .

- (ii) For all  $\mu > 0$  and  $K \in \mathbb{T}^d$  the operator  $H_\mu(K)$  has an eigenvalue  $E_\mu(K)$  lying above the top  $\tau_{\text{ess}}^t(H_\mu(K))$  of the essential spectrum. The eigenvalue  $E_\mu(K)$  is a holomorphic function in  $K \in \mathbb{T}^d$  and the associated eigenfunction (bound state)  $f_{\mu, E_\mu(K)}(\cdot, \cdot) \in L^{2,s}[(\mathbb{T}^d)^2]$  is holomorphic in  $(p, q) \in (\mathbb{T}^d)^2$ . Moreover, the vector valued mapping

$$f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dK); L^{2,s}[(\mathbb{T}^d)^2]], k \rightarrow f_{\mu, E_\mu(K)}$$

is also holomorphic in  $K \in \mathbb{T}^d$ .

- (iii) For all  $\mu < 0$  resp.  $\mu > 0$  and  $K \in \mathbb{T}^d$  the operator  $H_\mu(K)$  has no eigenvalue lying above the top resp. below the bottom of the essential spectrum.

Theorem 4.4 yields a corollary, which asserts the existence of a band spectrum for a system of two resp. three interacting bosons on the lattice  $\mathbb{Z}^d$ ,  $d = 1, 2$ .

**Corollary 4.5.** For any  $|\mu| > 0$  the two-particle resp. three-particle hamiltonian  $h_\mu$  resp.  $H_\mu$  has a band spectrum

$$[\min_k e_\mu(k), \max_k e_\mu(k)] \text{ resp. } [\min_K E_\mu(K), \max_K E_\mu(K)].$$

**Remark 4.6.** We remark that for large  $|\mu| > 0$  the two-particle resp. three-particle hamiltonian  $h_\mu$  resp.  $H_\mu$  has an isolated band

$$[\min_k e_\mu(k), \max_k e_\mu(k)] \text{ resp. } [\min_K E_\mu(K), \max_K E_\mu(K)].$$

Let

$$\sigma_{\text{esstwo}}(H_\mu(K)) = \cup_{k \in \mathbb{T}^d} \{e_\mu(K - k) + \varepsilon(k)\}$$

be the two-particle part and

$$\tau_{\text{ess}}^b(H_\mu(K)) = \inf \sigma_{\text{ess}}(H_\mu(K)) \text{ resp. } \tau_{\text{ess}}^t(H_\mu(K)) = \sup \sigma_{\text{ess}}(H_\mu(K))$$

be the bottom resp. top of the essential spectrum of the Schrödinger operator  $H_\mu(K)$ .

**Remark 4.7.** For any  $\mu < 0$  Theorems 4.2 and 4.3 yield that the relations

$$\sigma_{\text{esstwo}}(H_\mu(K)) \neq \emptyset$$

and hence

$$\tau_{\text{ess}}^b(H_\mu(K)) < E_{\min}(K)$$

hold, which allows the existence of bound states of three attractively interacting bosons on lattice  $\mathbb{Z}^d$  [2, 6].

**Remark 4.8.** We remark that for the three-particle Schrödinger operator  $H_\mu(K)$ ,  $K \in \mathbb{T}^3$  associated to a system of three bosons in the lattice  $\mathbb{Z}^3$ , there exists  $\mu_0 < 0$  such that

$$\tau_{\text{ess}}^b(H_{\mu_0}(0)) = E_{\min}(0)$$

and for all  $\mu < \mu_0 < 0$

$$(4.1) \quad \tau_{\text{ess}}^b(H_\mu(0)) < E_{\min}(0).$$



At the same time for any  $0 \neq K \in \mathbb{T}^3$  the following relations

$$\sigma_{\text{esstwo}}(H_{\mu_0}(K)) \neq \emptyset$$

and

$$(4.2) \quad \tau_{\text{ess}}^b(H_{\mu}(K)) < E_{\min}(K).$$

hold. Thus, only the operator  $H_{\mu_0}(0)$  has infinitely many number of eigenvalues below the bottom of the essential spectrum (the Efimov effect) [2, 6] and this result yields the existence of bound states of  $H_{\mu_0}(K)$ ,  $K \in \mathbb{G}_0$ , where  $\mathbb{G}_0 \subset \mathbb{T}^d$  is a neighborhood of the point  $0 \in \mathbb{T}^d$ . Moreover for any nonzero  $K \in \mathbb{T}^3$  the operator  $H_{\mu_0}(K)$  has only finitely many number of bound states.

**Remark 4.9.** Note that analogous remarks on the existence of bound states of  $H_{\mu}(K)$  for the case of repulsive  $\mu > 0$  are valid.

**Remark 4.10.** The results for the attractive interactions are characteristic to the Schrödinger operators associated to a system of three particles moving on the lattice  $\mathbb{Z}^d$  and the Euclidean space  $\mathbb{R}^d$  in dimension  $d = 1, 2$ .

## 5. THE ESSENTIAL SPECTRUM OF THE OPERATOR $H_{\mu}(K)$

Let  $\mu \neq 0$ . Since the particles are identical there is only one channel operator  $H_{\mu, ch}(K)$ ,  $K \in \mathbb{T}^d$ ,  $d = 1, 2$  defined in the Hilbert space  $L^2[(\mathbb{T}^d)^2]$  as

$$H_{\mu, ch}(K) = H_0(K) + \mu V.$$

The operators  $H_0(K)$  and  $V = V_{\alpha}$  act as follows

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L^2[(\mathbb{T}^d)^2],$$

where

$$E(K; p, q) = \varepsilon(K - p - q) + \varepsilon(q) + \varepsilon(p)$$

and

$$(Vf)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt), \quad f \in L^2[(\mathbb{T}^d)^2].$$

The decomposition of the space  $L^2[(\mathbb{T}^d)^2]$  into the direct integral

$$L^2[(\mathbb{T}^d)^2] = \int_{k \in \mathbb{T}^d} \oplus L^2(\mathbb{T}^d)\eta(dp)$$

yields for the operator  $H_{\mu, ch}(K)$  the decomposition

$$H_{\mu, ch}(K) = \int_{k \in \mathbb{T}^d} \oplus h_{\mu}(K, p)\eta(dp).$$

The fiber operator  $h_{\mu}(K, p)$  acts in the Hilbert space  $L^2(\mathbb{T}^d)$  and has the form

$$(5.1) \quad h_{\mu}(K, p) = h_{\mu}(K - p) + \varepsilon(p)I,$$

where  $I_{L^2(\mathbb{T}^d)}$  is the identity operator and the operator  $h_{\mu}(p)$  is unitarily equivalent to  $h_{\mu}(p)$ , defined by (3.3). The representation (5.1) of the operator  $h_{\mu}(K, p)$  and Theorem 4.2 yield the following description for the spectrum of  $h_{\mu}(K, p)$

$$(5.2) \quad \sigma(h_{\mu}(K, p)) = [e_{\mu}(K - p) + \varepsilon(p)] \cup [E_{\min}(K), E_{\min}(K)].$$

**Lemma 5.1.** *For any  $K \in \mathbb{T}^d$  the bottom  $\tau_{\text{ess}}^b(H_\mu(K))$  resp. top  $\tau_{\text{ess}}^t(H_\mu(K))$  of the essential spectrum satisfies the inequality*

$$\tau_{\text{ess}}^b(H_\mu(K)) < E_{\min}(K)$$

resp.

$$\tau_{\text{ess}}^t(H_\mu(K)) > E_{\max}(K).$$

*Proof.* For any  $\mu \neq 0$  and  $K \in \mathbb{T}^d$  we define  $Z_\mu(K, p)$  on  $\mathbb{T}^d$ ,  $d = 1, 2$  by

$$(5.3) \quad Z_\mu(K, p) = e_\mu(K - p) + \varepsilon(p).$$

Theorem 4.2 yields the inequality

$$\begin{aligned} Z_\mu(K, K - p_{\min}(K)) &= e_\mu(K - p_{\min}(K)) + \varepsilon(p_{\min}(K)) \\ &< \mathcal{E}_{\min}(K - p_{\min}(K)) + \varepsilon(p_{\min}(K)) = E_{\min}(K), \end{aligned}$$

where  $(p_{\min}(K), p_{\min}(K)) \in (\mathbb{T}^d)^2$  is a minimum point of the function  $E(K; p, q)$ . The definition of  $\tau_{\text{ess}}^b(H_\mu(K))$  gives

$$\tau_{\text{ess}}^b(H_\mu(K)) = \inf_{p \in \mathbb{T}^d} Z_\mu(K, p) \leq e_\mu(K - p_{\min}(K)) + \varepsilon(p_{\min}(K)) < E_{\min}(K),$$

which proves Lemma 5.1. For the case  $\mu > 0$  the proof of Lemma 5.1 is analogously.  $\square$

## 6. PROOF OF THE MAIN RESULTS

Set

$$\begin{aligned} E_{\min}(K, k) &= \min_q E(K, k; q) = \min_q \mathcal{E}_{K-k}(q) + \varepsilon(k), \\ E_{\max}(K, k) &= \max_q E(K, k; q) = \max_q \mathcal{E}_{K-k}(q) + \varepsilon(k). \end{aligned}$$

For any  $\mu \in \mathbb{R}$  and  $K, k \in \mathbb{T}^d$ ,  $d = 1, 2$  the determinant  $\Delta_\mu(K, k; z)$  associated to the operator  $h_\mu(K, k)$  can be defined as real-analytic function in  $C \setminus [E_{\min}(K, k), E_{\max}(K, k)]$  by

$$\Delta_\mu(K, k; z) = 1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{E(K; k, q) - z}.$$

**Lemma 6.1.** *For any  $\mu \in \mathbb{R}$  and  $K, k \in \mathbb{T}^d$  the number  $z \in C \setminus [E_{\min}(K, k), E_{\max}(K, k)]$  is an eigenvalue of the operator  $h_\mu(K, k)$  if and only if*

$$\Delta_\mu(K, k; z) = 0.$$

The proof of Lemma 6.1 is simple and can be found in [5].

**Remark 6.2.** *We note that for each  $\mu < 0$  resp.  $\mu > 0$  and  $K, k \in \mathbb{T}^d$  there exist  $z_1 = z_1(K, k) < E_{\min}(K, k)$  resp.  $z_2 = z_2(K, k) > E_{\max}(K, k)$  such that for all  $z \leq z_1$  resp.  $z \geq z_2$  the function  $\Delta_\mu(K, k; z)$  is non-negative and the square root function  $\Delta_\mu^{\frac{1}{2}}(K, k; z)$  is well defined.*

We define for each  $\mu \in \mathbb{R}$  and  $z \in \mathbb{R} \setminus [\tau_{\text{ess}}^b(H_\mu(K)), \tau_{\text{ess}}^t(H_\mu(K))]$  the self-adjoint compact operator  $L_\mu(K, z)$ ,  $K \in \mathbb{T}^d$  as

$$(6.1) \quad [L_\mu(K, z)\psi](p) = 2\mu \int_{\mathbb{T}^d} \frac{\Delta_\mu^{-\frac{1}{2}}(K, p, z) \Delta_\mu^{-\frac{1}{2}}(K, q, z)}{E(K; p, q) - z} \psi(q) \eta(dq), \psi \in L^2(\mathbb{T}^d).$$

Notice that for  $\mu < 0$  the operator  $L_\mu(K, z)$ ,  $z < \tau_{\text{ess}}^b(H_\mu(K))$  has been introduced in [6] to investigate Efimov's effect for the three-particle lattice Schrödinger operator  $H_\mu(K)$  associated to a system of three bosons on the lattice  $\mathbb{Z}^3$ .

**Lemma 6.3.** *Let  $\mu \neq 0$  and  $z \in \mathbb{R} \setminus [\tau_{\text{ess}}^b(H_\mu(K)), \tau_{\text{ess}}^t(H_\mu(K))]$ .*

(i) *If  $f \in L^{2,s}[(\mathbb{T}^d)^2]$  solves  $H_\mu(K)f = zf$ , then*

$$\psi(p) = \Delta_\mu^{\frac{1}{2}}(K, p; z) \int_{\mathbb{T}^d} f(p, t) \eta(dt) \in L^2(\mathbb{T}^d)$$

*solves  $L_\mu(K, z)\psi = -\psi$ .*

(ii) *If  $\psi \in L^2(\mathbb{T}^d)$  solves  $L_\mu(K, z)\psi = -\psi$ , then*

$$f(p, q) = \frac{-\mu[\varphi(p) + \varphi(q) + \varphi(K - p - q)]}{E(K; p, q) - z} \in L^{2,s}[(\mathbb{T}^d)^2]$$

*solves the equation  $H_\mu(K)f = zf$ , where  $\varphi(p) = \Delta_\mu^{-\frac{1}{2}}(K, p; z)\psi(p)$ .*

*Proof.*

(i) We prove Lemma 6.3 for the case  $\mu < 0$  and  $z < \tau_{\text{ess}}^b(H_\mu(K))$ . The case  $\mu > 0$  and  $z > \tau_{\text{ess}}^t(H_\mu(K))$  can be proven analogously.

(i) Let  $\mu < 0$ . Assume that for some  $K \in \mathbb{T}^d$  and  $z < \tau_{\text{ess}}^b(H_\mu(K))$  the Schrödinger equation

$$(H_\mu(K)f)(p, q) = zf(p, q),$$

i.e., the equation

$$(6.2) \quad [E(K; p, q) - z]f(p, q) = -\mu \left[ \int_{\mathbb{T}^d} f(p, t) \eta(dt) + \int_{\mathbb{T}^d} f(t, q) \eta(dt) + \int_{\mathbb{T}^d} f(K - p - q, t) \eta(dt) \right]$$

has solution  $f \in L^{2,s}[(\mathbb{T}^d)^2]$ . Denoting by  $\varphi(p) = \int_{\mathbb{T}^d} f(p, t) \eta(dt) \in L^2(\mathbb{T}^d)$  we rewrite equation (6.2) as follows

$$(6.3) \quad f(p, q) = -\mu \frac{\varphi(p) + \varphi(q) + \varphi(K - p - q)}{E(K; p, q) - z},$$

which gives for  $\varphi \in L^2(\mathbb{T}^d)$  the equation

$$(6.4) \quad \varphi(p) = -\mu \int_{\mathbb{T}^d} \frac{\varphi(p) + \varphi(t) + \varphi(K - p - t)}{E(K; p, t) - z} \eta(dt).$$

Since the function  $E(K; p, t)$  is invariant under the changing  $K - p - t \rightarrow t$  of variables we have

$$\varphi(p) \left[ 1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{E(K; p, q) - z} \right] = -2\mu \int_{\mathbb{T}^d} \frac{\varphi(q)}{E(K; p, q) - z} \eta(dq).$$

Denoting by  $\psi(p) = \Delta_\mu^{\frac{1}{2}}(K, p; z)\varphi(p)$  and taking into account that  $\Delta_\mu(K, p; z) \neq 0$ ,  $z < \tau_{\text{ess}}^b(H_\mu(K))$  we get the equation

$$(6.5) \quad L_\mu(K, z)\psi = -\psi$$

(ii) Assume that  $\psi$  is a solution of equation (6.5). Then the function

$$(6.6) \quad \varphi(p) = \Delta_\mu^{-\frac{1}{2}}(K, p; z)\psi(p)$$

is a solution of equation (6.4) and hence the function defined by (6.3) is a solution of the equation  $H_\mu(K)f = zf$ , i.e., is an eigenfunction of the operator  $H_\mu(K)$  associated to the eigenvalue  $z < \tau_{\text{ess}}^b(H_\mu(K))$ .  $\square$

**Proof of Theorem 4.3.** Theorem 4.3 can be proven, applying equality (5.2), by the same way as Theorem 3.2 in [1, 2].  $\square$

**Proof of Theorem 4.4. (i)**

Let  $\mu < 0$  and  $L_\mu(K, z), K \in \mathbb{T}^d, d = 1, 2$  be the operator defined in (6.1). Then for any non-zero  $f \in L^2(\mathbb{T}^d)$  and  $z < \tau_{\text{ess}}^b(H_\mu(K))$  the following relations

$$(6.7) \quad \begin{aligned} (L_\mu(K, z)f, f) &= -2\mu \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{f(p)\overline{f(q)}\eta(dp)\eta(dq)}{\Delta_\mu^{\frac{1}{2}}(K, p, z)\Delta_\mu^{\frac{1}{2}}(K, q, z)(E(K; p, q) - z)} \\ &> \frac{-2\mu}{E_{\max} - z} \left| \int_{\mathbb{T}^d} \frac{f(p)\eta(dp)}{\Delta_\mu^{\frac{1}{2}}(K, p, z)} \right|^2 \geq 0 \end{aligned}$$

hold. Let

$$F_z(f) = \int_{\mathbb{T}^d} \frac{f(p)\eta(dp)}{\Delta_\mu^{\frac{1}{2}}(K, p, z)}, \quad z < \tau_{\text{ess}}^b(H_\mu(K))$$

be bounded linear functional defined on  $L^2(\mathbb{T}^d)$ . According the Riesz theorem

$$\|F_z\| = \left[ \int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p, z)} \right]^{\frac{1}{2}}.$$

Let  $\mathfrak{M}_+ \subset L^2(\mathbb{T}^d)$  subset of all non-negative functions. Then

$$(6.8) \quad \|F_z\| = \sup_{f \in L^2(\mathbb{T}^d)} \frac{|F_z(f)|}{\|f\|} \geq \sup_{f \in \mathfrak{M}_+} \frac{|F_z(f)|}{\|f\|} \geq \frac{|F_z(\psi_z)|}{\|\psi_z\|} = \|\psi_z\|,$$

where  $\psi_z = [\Delta_\mu^{\frac{1}{2}}(K, p, z)]^{-1}$ . Hence we have that  $\|F_z\| = \|\psi_z\|$ .

Since for any  $p \in \mathbb{T}^d$  the function  $\Delta_\mu(K, p, z)$  is monotone decreasing in  $z \in (-\infty, \tau_{\text{ess}}^b(H_\mu(K)))$  there exists a.e. the point-wise limit

$$\lim_{z \rightarrow \tau_{\text{ess}}^b(H_\mu(K))} \frac{1}{\Delta_\mu(K, p; z)} = \frac{1}{\Delta_\mu(K, p; \tau_{\text{ess}}^b(H_\mu(K)))}.$$

The Fatou's theorem yields the inequality

$$\int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p; \tau_{\text{ess}}^b(H_\mu(K)))} \leq \liminf_{z \rightarrow \tau_{\text{ess}}^b(H_\mu(K))} \int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p; z)}.$$

Let  $p_\mu(K) \in \mathbb{T}^d, K \in \mathbb{T}^d$  be a minimum point of the function  $Z_\mu(K, p), K \in \mathbb{T}^d$  defined in (5.3). Then  $Z_\mu(K, p)$  has the following asymptotics

$$(6.9) \quad Z_\mu(K, p) = \tau_{\text{ess}}^b(H_\mu(K)) + (B(K)(p - p_\mu(K)), p - p_\mu(K)) + o(|p - p_\mu(K)|^2),$$

as  $|p - p_\mu(K)| \rightarrow 0$ , where  $B(K)$  is non-negative matrix. For any  $K, p \in \mathbb{T}^d$  there exists a  $\gamma = \gamma(K, p) > 0$  neighborhood  $W_\gamma(Z_\mu(K, p))$  of the point  $Z_\mu(K, p) \in \mathbb{C}$  such that for

all  $z \in W_\gamma(Z_\mu(K, p))$  the following equality holds

$$(6.10) \quad \Delta_\mu(K, p, z) = \sum_{n=1}^{\infty} C_n(\mu, K, p)[z - Z_\mu(K, p)]^n,$$

where

$$C_1(\mu, K, p) = -\mu \int_{\mathbb{T}^d} \frac{\eta(dq)}{[E(K; p, q) - Z_\mu(K, p)]^2} > 0.$$

According to (6.10) for any  $K \in \mathbb{T}^d$  there is  $U_{\delta(K)}(p_\mu(K))$  such that for all  $p \in U_{\delta(K)}(p_\mu(K))$  the equality

$$(6.11) \quad \begin{aligned} \Delta_\mu(K, p, \tau_{\text{ess}}^b(H_\mu(K))) \\ = (Z_\mu(K, p) - \tau_{\text{ess}}^b(H_\mu(K)))\hat{\Delta}_\mu(K, p, \tau_{\text{ess}}^b(H_\mu(K))) \end{aligned}$$

holds. Putting (6.9) into (6.11) yields the estimate

$$\Delta_\mu(K, p, \tau_{\text{ess}}^b(H_\mu(K))) \leq M(K)|p - p_\mu(K)|^2.$$

Hence, we have

$$\int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p; \tau_{\text{ess}}^b(H_\mu(K)))} = +\infty.$$

Consequently, for any  $P > 0$  there exists  $z_0 < \tau_{\text{ess}}^b(H_\mu(K))$  such that the inequality

$$(6.12) \quad \|F_{z_0}\| = \sup_{\|\psi\|=1} (F_{z_0}\psi, \psi) = \left[ \int_{\mathbb{T}^d} \frac{\eta(dp)}{\Delta_\mu(K, p, z)} \right]^{\frac{1}{2}} > P$$

holds. Since for all  $z \leq \tau_{\text{ess}}^b(H_\mu(K))$  the positive function  $(E_{\max} - z)^{-1}$  is bounded above, the inequalities (6.7) and (6.12) yield the existence  $\psi \in L^2(\mathbb{T}^d)$ ,  $\|\psi\|_{L^2(\mathbb{T}^d)} = 1$  such that the relation  $(L_\mu(K, z_0)\psi, \psi) > 1$  holds. At the same time  $(L_\mu(K, z)\psi, \psi)$  is continuous in  $z \in (-\infty, \tau_{\text{ess}}^b(H_\mu(K)))$  and

$$(L_\mu(K, z)\psi, \psi) \rightarrow 0 \text{ as } z \rightarrow -\infty.$$

Thus, there exists real number  $E_\mu(K)$ ,  $-\infty < E_\mu(K) < z_0 < \tau_{\text{ess}}^b(H_\mu(K))$  that satisfies the equality

$$(L_\mu(K, E_\mu(K))\psi, \psi) = 1$$

and hence the Hilbert-Schmidt theorem implies that the equation

$$L_\mu(K, E_\mu(K))\psi = \psi$$

has a solution  $\psi \in L^2(\mathbb{T}^d)$ ,  $\|\psi\| = 1$ .

Lemma 6.3 yields that  $E_\mu(K) < \tau_{\text{ess}}^b(H_\mu(K))$  is an eigenvalue of the operator  $H_\mu(K)$  and the associated eigenfunction  $f_{E_\mu(K)}(K; p, q)$  takes the form

$$(6.13) \quad f_{E_\mu(K)}(K; p, q) = \frac{-\mu c(K)[\varphi(p) + \varphi(q) + \varphi(K - p - q)]}{E(K; p, q) - E_\mu(K)} \in L^{2,s}[(\mathbb{T}^d)^2]$$

with  $c(K) = \|f_{E_\mu(K)}(K; p, q)\|^{-1}$ ,  $K \in \mathbb{T}^d$  being the normalizing constant.

Since for any  $K \in \mathbb{T}^d$  the functions  $\Delta_\mu(K, p; E_\mu(K))$  and  $E(K; p, q) - E_\mu(K) > 0$  are holomorphic in  $p, q \in \mathbb{T}^d$  the solution  $\psi$  of equation (6.5) and the function  $\varphi$  given in (6.6) are holomorphic in  $p \in \mathbb{T}^d$ . Hence, the eigenfunction (6.13) of the operator  $H_\mu(K)$  associated to the eigenvalue  $E_\mu(K) < \tau_{\text{ess}}^b(H_\mu(K))$  is also holomorphic in  $p, q \in \mathbb{T}^d$ .

For any  $z < \tau_{\text{ess}}^b(H_\mu(K))$  the kernel function

$$L_\mu(K, z; p, q) = -2\mu \frac{\Delta_\mu^{-\frac{1}{2}}(K, p, z) \Delta_\mu^{-\frac{1}{2}}(K, q, z)}{E(K; p, q) - z}$$

of the compact self-adjoint operator  $L_\mu(K, z)$  is holomorphic in  $p, q \in \mathbb{T}^d$ . The Fredholm determinant  $D_\mu(K, z) = \det[I - L_\mu(K, z)]$  associated to the kernel function is real-analytic function in  $z \in (-\infty, \tau_{\text{ess}}^b(H_\mu(K)))$ . Lemma 6.3 and the Fredholm theorem yield that each eigenvalue of the operator  $H_\mu(K)$  is a zero of the determinant  $D_\mu(K, z)$  and vice versa. Consequently, the compactness of the torus  $\mathbb{T}^d$  and the implicit function theorem give that the eigenvalue  $E_\mu(K)$  of  $H_\mu(K)$  is a holomorphic function in  $K \in \mathbb{T}^d$ ,  $d = 1, 2$ .

Since for any  $p, q \in \mathbb{T}^d$  the functions  $\Delta_\mu(K, p; E_\mu(K))$  and  $E(K; p, q) - E_\mu(K)$  are holomorphic in  $K \in \mathbb{T}^d$  the solution  $\psi$  of (6.5) and the function  $\varphi$  defined by (6.6) are holomorphic in  $K \in \mathbb{T}^d$ . Hence, the eigenfunction (6.13) of the operator  $H_\mu(K)$  associated to the eigenvalue  $E_\mu(K) < \tau_{\text{ess}}^b(H_\mu(K))$  is also holomorphic in  $K \in \mathbb{T}^d$ . Consequently, the vector valued mapping

$$f_\mu : \mathbb{T}^d \rightarrow L^2[\mathbb{T}^d, \eta(dK); L^{2,s}[(\mathbb{T}^d)^2]], K \rightarrow f_{\mu,K}(\cdot, \cdot)$$

is holomorphic in  $\mathbb{T}^d$ .

(ii) This part of Theorem 4.4 can be proven similarly using the corresponding Lemmas.

(iii) An application the minimax principle to the operator  $H_\mu(K)$  completes the proof of Theorem 4.4.

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